Global Well-posedness for the Critical Dissipative Quasi-Geostrophic Equations in L^∞

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Abstract

In this paper, we study the critical dissipative quasi-geostrophic equations in scaling invariant spaces. We prove that there exists a global in time solution for small data $\theta_0 \in L^\infty \cap \dot{H}^1$ such that $\mathscr{R}(\theta_0) \in L^\infty$, where $\mathscr{R}$ is a Riesz transform. As a corollary, we prove that if in addition, $\theta_0 \in \dot{B}^0_{\infty,q}$, $1 \le q < 2$, is small enough, then $\theta \in \tilde{L}^\infty_t \dot{B}^0_{\infty,q} \cap \tilde{L}^1_t \dot{B}^1_{\infty,q}$.

1 Introduction

In this paper, we are concerned with the dissipative quasi-geostrophic equations in two dimensions. These equations are derived from the more general quasi-geostrophic approximation for nonhomogeneous fluid flow in a rapidly rotating three-dimensional half-space with small Rossby and Ekman numbers. The system of equations in two dimensions is given by

$$(DQG)\begin{cases} \theta_t + v\cdot\nabla\theta + \kappa(-\triangle)^\alpha\theta = 0 \\ v = (-R_2\theta, R_1\theta) \end{cases}$$

where the scalar function θ represents the potential temperature, v is the fluid velocity, and $0 \le \alpha \le 1$. $(-\triangle)^\alpha$ is a pseudo-differential operator and we denote by $\Lambda^{2\alpha}$, which means that $\mathscr{F}(\Lambda^{2\alpha} f) = |\xi|^{2\alpha}\mathscr{F}(f)$. Here, R_1, R_2 are the usual Riesz transforms:

$$R_l f(x) = \mathscr{F}^{-1}(\frac{i\xi_l}{|\xi|}\hat{f}(\xi))(x), \quad l = 1, 2$$

For the simplicity, we take $\kappa = 1$. The cases $\alpha > \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\alpha < \frac{1}{2}$ are called respectively sub-critical, critical and super-critical.

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The critical quasi-geostrophic equations are the dimensionally correct analogue to the three dimensional Navier-Stokes equations. In two dimensions, the Navier-Stokes equations are globally well-posed, while the regularity problems for the Navier-Stokes equations in three dimensions are still open. However, the regularity problems for the critical quasi-geostrophic equations are solved recently by two groups. Caffarelli-Vasseur [3] proved the global Holder regularity to the critical quasi-geostrophic equations. They use harmonic extension to prove a gain of regularity of weak solutions. Kiselev-Nazarov-Volberg [14] also proved that the critical quasi-geostrophic equation with periodic smooth initial data θ_0 has a unique global smooth solution by using the modulus of continuity argument.

In this paper, we consider (DQG) in scaling invariant spaces. Solvability of evolution equations in scaling invariant spaces are well-developed in the context of the Navier-Stokes equations. For example, if we restrict function spaces to the energy spaces, the optimal result is due to Fujita-Kato in [11]. Later, Chemin [6] proved similar results in the framework of Besov spaces, $\dot{B}^{\frac{d}{p}-1}_{p,q}$. Let us obtain the scaling invariant critical spaces for (DQG). Since the equation is invariant under the following scaling,

$$\theta_\lambda(t,x) = \lambda^{2\alpha-1}\theta(\lambda^{2\alpha}t, \lambda x), \text{ with initial data } \theta_\lambda(0,x) = \theta_{\lambda 0}(x) = \lambda^{2\alpha-1}\theta_0(\lambda x)$$

$\dot{B}^{1+\frac{2}{p}-2\alpha}_{p,q}$ are critical spaces in Besov spaces. Let $s_p = 1 + \frac{2}{p} - 2\alpha$. The cases $s > s_p$, $s = s_p$, and $s < s_p$ are called respectively super-critical, critical and sub-critical. For global in time results with small initial data in critical Besov spaces with $p < \infty$, see [2], [4], [7], [13], [19], and [20] and references therein. Finally, Dong-Du [10] proved the global wellposedness result for large data in H^1.

For $p = \infty$ and $\alpha = \frac{1}{2}$, L^∞ is a critical space in Lebesgue spaces. However, it seems that L^∞ is not a suitable space for the problem because the velocity field v is an image of the singular integral operator so that v is in BMO, not in L^∞. Instead, people have studied the problem in Besov spaces $\dot{B}^0_{\infty,1}$ because the Riesz transforms commute with dyadic operators $\triangle_j$. ([1], [12])

In this paper, we deal with the critical dissipative quasi-geostrophic equations in L^∞ with some additional conditions on initial data. As mentioned earlier, the main difficulty is that v is in BMO if θ is in L^∞. Moreover, to estimate the velocity in L^∞ in terms of θ in energy space H^s, then s should be strictly bigger than 1. Nevertheless, we can obtain a solution in the limiting space $L^\infty \cap \dot{H}^1$. The main result of the paper is the following.

Theorem 1.1 *There exists a constant $\epsilon_0 > 0$ such that for any $\theta_0 \in L^\infty$ with*

$$||\theta_0||_{L^\infty} + ||v_0||_{L^\infty} + ||\theta_0||_{\dot{H}^1} < \epsilon_0$$

the critical quasi-geostrophic equation has a unique solution θ, which belongs to $L^\infty_t(L^\infty_x \cap \dot{H}^1_x)$.

Remark 1 *Dong-Li [10] proved that there exists a global solution for large data in* H^1. *In this paper, we only focus on the persistence the* L^∞ *norm of the solution so that we solve the problem globally in time for small data in* $\dot{H}^1$.

Corollary 1.2 *Under the condition of Theorem 1.1, if in addition,* $\theta_0 \in \dot{B}^0_{\infty,q}$, $1 \le q < 2$, *is small enough, then* $\theta \in \tilde{L}^\infty_t \dot{B}^0_{\infty,q} \cap \tilde{L}^1_t \dot{B}^1_{\infty,q}$.

Remark 2 *Recently, Wang-Zhang [18] proved that there exists a global in time solution for the critical quasi-geostrophic equations for large data in inhomogeneous spaces* $B^0_{\infty,q}$, $q < \infty$, *and for small data in* $B^0_{\infty,\infty}$. *The reason why they take inhomogeneous spaces is that homogeneous spaces behave badly in negative indices. In our paper, we prove the global well-posedness for small data in homogeneous spaces* $\dot{B}^0_{\infty,q}$, $1 \le q < 2$. *We take the range of* q *in* $[1, 2)$ *to make* $\dot{H}^1 \nsubseteq \dot{B}^0_{\infty,q}$ *and* $\dot{B}^0_{\infty,q} \nsubseteq \dot{H}^1$. *We still do not know, however, whether or not the solution is globally defined for large data.*

Remark 3 *In this paper, we strongly relied on the* $\dot{H}^1$ *estimate to take* $\nabla\theta$ *in* L^4. *Then, we can take the Oseen kernel* $L^{\frac{4}{3}}$ *in space. (As one can see, the Oseen kernel is not integral.) But, we do not know how to solve the problem without the assumption that* $\theta_0 \in \dot{H}^1$.

Notations: • $P_t = \dfrac{1}{t^2} P(\dfrac{x}{t})$, $\quad P(x) = \dfrac{C}{(1+|x|^2)^{\frac{3}{2}}}$ is the Poisson kernel in two dimensions. We also denote it by $e^{-\Lambda t}$.

• $L^p_t X$ denotes a function space, which is defined on time interval $(0, \infty)$ with values in a Banach space X.

• $A \lesssim B$ means there is a constant C such that $A \le CB$.

2 Preliminaries

2.1 Littlewood-Paley Decomposition

We take a couple of smooth functions (χ, ϕ) supported on $\{\xi; |\xi| \le 1\}$ with values in $[0, 1]$ such that for all $\xi \in R^d$,

$$\chi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1$$

where $\psi(\xi) = \phi(\frac{\xi}{2}) - \phi(\xi)$. We denote $\psi(2^{-j}\xi)$ by $\psi_j(\xi)$. This is called the Littlewood-Paley dyadic partition of unity. We apply this decomposition to elements in $\mathscr{S}'$. Let

$$\triangle_j u = 0 \quad \text{if} \quad j \le -2 \quad \triangle_{-1} u = \chi(D) u = \tilde{h} \star u \quad \text{with} \quad \tilde{h} = \mathscr{F}^{-1}\chi$$

$$\triangle_j u = \psi_j(D)u = 2^{jd}\int h(2^j y)u(x-y)dy \quad \text{with} \quad h = \mathscr{F}^{-1}\psi, \quad \text{if} \quad j \geq 0$$

We define a nonhomogeneous Littlewood-Paley decomposition as follows.

$$u = \sum_{j=-1}^{\infty} \triangle_j u, \quad u \in \mathscr{S}'$$

We also define a homogeneous Littlewood-Paley decomposition. We take a couple of smooth functions (χ, ϕ) as before. We define homogeneous dyadic blocks by

$$\dot{\triangle} u = \psi(2^{-j}D)u \quad \text{for all} \quad j \in \mathbb{Z}, \quad \dot{S}_j u = \chi(2^{-j}D)u \quad \text{for all} \quad j \in \mathbb{Z}$$

Then, for $u \in \mathscr{S}'$, we have $u = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j u$ modulo a polynomial only.

2.2 Besov Spaces

Let $s \in R$, $p, q \in [1, \infty]$. Then the inhomogeneous and homogeneous Besov semi-norms are defined, respectively, by

$$||u||_{B^s_{p,q}} = ||S_0 u||_{L^p} + (\sum_{j\geq -1} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}, \quad ||u||_{\dot{B}^s_{p,q}} = (\sum_{j\in\mathbb{Z}} 2^{qjs}||\triangle_j u||^q_{L^P})^{\frac{1}{q}}$$

If u is time-dependent,

$$||u||_{L^\rho_t B^s_{p,q}} = ||S_0 u||_{L^\rho_t L^p} + ||(\sum_{j\geq -1} 2^{qjs}||\triangle_j u||^q_{L^p})^{\frac{1}{q}}||_{L^\rho_t}, \quad ||u||_{L^\rho_t \dot{B}^s_{p,q}} = ||(\sum_{j\in\mathbb{Z}} 2^{qjs}||\triangle_j u||^q_{L^p})^{\frac{1}{q}}||_{L^\rho_t}$$

By changing the order of time integration and the summation, we define semi-norms as

$$||u||_{\tilde{L}^\rho_t B^s_{p,q}} = ||S_0 u||_{L^\rho_t L^p} + (\sum_{j\geq -1} 2^{qjs}||\triangle_j u||^q_{L^\rho_t L^p})^{\frac{1}{q}}, \quad ||u||_{\tilde{L}^\rho_t \dot{B}^s_{p,q}} = (\sum_{j\in\mathbb{Z}} 2^{qjs}||\triangle_j u||^q_{L^\rho_t L^p})^{\frac{1}{q}}$$

By changing these orders, one can avoid the time singularity at the origin of the heat kernel when we estimate the solution in the integral form in chapter 4. According to Minkowski inequality, we have

$$||u||_{\tilde{L}^\rho_t B^s_{p,q}} \leq ||u||_{L^\rho_t B^s_{p,q}} \quad \text{if} \quad \rho \leq q, \quad ||u||_{\tilde{L}^\rho_t B^s_{p,q}} \geq ||u||_{L^\rho_t B^s_{p,q}} \quad \text{if} \quad \rho \geq q$$

The main reason of decomposing a tempered distribution into a sum of dyadic blocks $\triangle_j u$, whose support in Fourier space is localized in a corona, is that we can apply the Sobolev embedding to each dyadic blocks. This fact is illustrated by the following Bernstein's lemma. For its proof, see [5].

Lemma 2.1 *(a) Let $1 \leq p \leq q \leq \infty$, $k \in \mathbb{N}$, and $R > 0$. For $f \in \mathscr{S}'$ whose Fourier transform $\hat{f}$ is supported in the ball $|\xi| \lesssim \lambda R$,*

$$\sup_{|\alpha|=k} ||\partial^\alpha f||_{L^p} \lesssim \lambda^k ||f||_{L^p}, \quad ||f||_{L^q} \lesssim \lambda^{d(\frac{1}{p}-\frac{1}{q})} ||f||_{L^p}$$

(b) For $f \in \mathscr{S}'$ whose Fourier transform $\hat{f}$ is supported in the corona $|\xi| \sim \lambda R$,

$$\sup_{|\alpha|=k} ||\partial^\alpha f||_{L^p} \simeq \lambda^k ||f||_{L^p}, \quad ||f||_{L^q} \lesssim \lambda^{d(\frac{1}{p}-\frac{1}{q})} ||f||_{L^p}$$

From this Lemma, we can prove the following two embedding properties, which we will use in chapter 3 and chapter 4.

$$\dot{B}^{\frac{d}{p}}_{p,1} \subset L^\infty, \quad \dot{B}^{\frac{d}{p}-1}_{p,1} \subset L^d \text{ for } p < d$$

The fundamental idea of the paper [6] is to localize the heat equation and estimate each dyadic block in $L^\rho_t L^p_x$. Then, one can extract maximal regularities in L^1 in time from the heat kernel. We need the same estimates for fractional Laplacians. These are proved in [12]. They imitated the same idea in [6].

Lemma 2.2 *Let $\mathscr{C}$ be a ring. There exists a positive constant C such that for any $p \in [1, \infty]$, for any couple (t, λ) of positive real numbers, we have*

$$||e^{t\triangle} u||_{L^p} \leq C e^{-t(\lambda)^2} ||u||_{L^p} \text{ for supp } \hat{u} \in \lambda\mathscr{C}$$

2.3 Paraproduct

The concept of paraproduct is to deal with the interaction of two functions in terms of low or high frequency parts. For u, v two tempered distributions, we have the formal decomposition:

$$uv = \sum_{i,j} \triangle_i u \triangle_j v$$

Definition 2.3 *Let u and v be two tempered distributions. Then,*

$$uv = T_u v + T_v u + R(u, v)$$

$$T_u v = \sum_{i \leq j-2} \triangle_i u \triangle_j v = \sum_{j \in \mathbb{Z}} S_{j-1} u \triangle_j v, \quad R(u, v) = \sum_{|j-j'| \leq 1} \triangle_j u \triangle_{j'} v$$

where $S_j u = \sum_{l \leq j-1} \triangle_l u$. We list some continuity properties for the inhomogeneous paraproduct and the remainder. The reader is referred to [9] for more results on the subject.

Proposition 2.4 *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.*
(i) T is a bilinear continuous operator from $L^\infty \times B^s_{p,q}$ to $B^s_{p,q}$ such that

$$||T||_{\mathscr{L}(L^\infty \times B^s_{p,q} \to B^s_{p,q})} \lesssim C^{|s|+1}$$

(ii) Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$. Assume that

$$\tfrac{1}{p} \leq \tfrac{1}{p_1} + \tfrac{1}{p_2} \leq 1, \quad \tfrac{1}{q} \leq \tfrac{1}{q_1} + \tfrac{1}{q_2}, \quad s_1 + s_2 > 0$$

Then, the remainder R maps $B^{s_1}_{p_1,q_1} \times B^{s_2}_{p_2,q_2}$ to $B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,q}$ such that

$$||R(u,v)||_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,q}} \lesssim \frac{C^{|s_1+s_2|+1}}{s_1+s_2} ||u||_{B^{s_1}_{p_1,q_1}} ||v||_{B^{s_2}_{p_2,q_2}}$$

(iii) For $s > 0$, $B^s_{p,q} \cap L^\infty$ is an algebra and

$$||uv||_{B^s_{p,q}} \lesssim ||u||_{L^\infty} ||v||_{B^s_{p,q}} + ||v||_{L^\infty} ||u||_{B^s_{p,q}}$$

2.4 Auxiliary lemmas

If one takes the $L^p - L^q$ type estimates to the Poisson kernel, then, the heat kernel generates the time singularity near the origin appears. To control this singularity in terms of time integration, we need the following Lemma.

Lemma 2.5 *Hardy-Littlewood-Sobolev Inequality [16]: Let $0 < \lambda < d$, $\dfrac{1}{p} + \dfrac{\lambda}{d} + \dfrac{1}{q} = 2$. Then,*

$$\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \frac{1}{|x-y|^\lambda} g(y) dy dx \Big| \lesssim ||f||_{L^p} ||g||_{L^q}$$

In particular, for the one dimensional case,

$$\Big| \int_0^t \frac{1}{|t-s|^{\frac{1}{2}}} a(s) ds \Big| \lesssim ||a||_{L^2}$$

In this paper, we need to deal with the Riesz transform acting on the Poisson kernel. In fact, we have a nice representation of such a kernel as follows.

Lemma 2.6 *Oseen Kernel [15]: The operator $O_t = \mathscr{R} K_t$ is a convolution operator $O_t f = \tilde{K}_t \star f$, where the kernel $\tilde{K}_t$ satisfies*

$$\tilde{K}_t(x) = \frac{1}{t^d} K(\frac{x}{t})$$

for a smooth function $\tilde{K}$ such that for all $\alpha \in \mathbb{N}^d$,

$$(1+|x|)^{d+\alpha}\partial^\alpha \tilde{K} \in L^\infty$$

Here, $\mathscr{R}$ is a Riesz transform. In particular, $\tilde{K}$ is in L^p for $p > 1$.

Finally, we provide the L^p estimate of the Poisson kernel and its image under the Riesz transform.

Lemma 2.7 *For any $1 < p < \infty$,*

$$||P_t||_{L^p} \lesssim t^{2(1-\frac{1}{p})}, \quad ||\mathscr{R}P_t||_{L^p} \lesssim t^{2(1-\frac{1}{p})}$$

3 Proof of Theorem 1.1

In this chapter, we only obtain a priori estimate in $L^\infty_t L^\infty \cap L^\infty_t \dot{H}^1$. After that, we will briefly present how to construct a solution by iteration scheme. For the details, see [2], [4].

3.1 A priori estimate

First, we estimate the solution in $\dot{H}^1$. By taking one derivative to the equation,

$$\partial\theta_t + v\cdot\nabla\partial\theta + \partial v\cdot\nabla\theta - \Lambda\partial\theta = 0 \tag{1}$$

We multiply by $\partial\theta$ and do the energy estimate. Then,

$$\frac{1}{2}\frac{d}{dt}||\partial\theta||^2_{L^2} + ||\partial^{\frac{1}{2}}\nabla\theta||^2_{L^2} \le \int |\partial v||\nabla\theta||\partial v|dx \lesssim ||\nabla\theta||_{L^2}||\nabla v||^2_{L^4} \tag{2}$$

By Sobolev embedding $\dot{H}^{\frac{1}{2}} \subset L^4$,

$$\frac{d}{dt}||\partial\theta||^2_{L^2} + ||\partial^{\frac{1}{2}}\nabla\theta||^2_{L^2} \lesssim ||\nabla\theta||_{L^2}||\partial^{\frac{1}{2}}\nabla v||^2_{L^2} \tag{3}$$

Since Riesz transforms map H^s to H^s,

$$\frac{d}{dt}||\partial\theta||^2_{L^2} + ||\partial^{\frac{1}{2}}\nabla\theta||^2_{L^2} \lesssim ||\nabla\theta||_{L^2}||\partial^{\frac{1}{2}}\nabla \theta||^2_{L^2} \tag{4}$$

This implies that there exists a global-in time solution θ in $L^\infty_t\dot{H}^1 \cap L^2_t\dot{H}^{\frac{3}{2}}$ if initial data θ_0 is sufficiently small in $\dot{H}^1$.

Now, we estimate the solution in L^∞. We represent the solution as the integral form:

$$\theta(t) = P_t \star \theta_0 - \int_0^t P_{t-s}\star(v\cdot\nabla\theta)(s)ds \tag{5}$$

where $\star$ is the convolution operator in space variables. By taking L^∞ in space,

$$\begin{aligned} ||\theta(t)||_{L^\infty} &\leq ||\theta_0||_{L^\infty} + \int_0^t ||P_{t-s} \star (v \cdot \nabla\theta)(s)||_{L^\infty} ds \\ &\lesssim ||\theta_0||_{L^\infty} + \int_0^t ||P_{t-s}||_{L^{\frac{4}{3}}} ||v(s)||_{L^\infty} ||\nabla\theta(s)||_{L^4} ds \\ &\lesssim ||\theta_0||_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-s}} ||v(s)||_{L^\infty} ||\nabla\theta(s)||_{L^4} ds \\ &\lesssim ||\theta_0||_{L^\infty} + ||v||_{L_t^\infty L^\infty} ||\nabla\theta||_{L_t^2 L^4} \end{aligned}$$

where we use Hardy-Littlewood-Sobolev inequality for the last inequality. As before, we can estimate $||\nabla\theta||_{L_t^2 L^4}$ in terms of $||\partial^{\frac{1}{2}}\nabla\theta||_{L_t^2 L^2}$ by the Sobolev embedding, which is small by the $\dot{H}^1$ estimate above. Since Riesz transforms do not map L^∞ to L^∞, we have to estimate v in L^∞ separately. We take a Riesz transform $\mathscr{R}$ to (5).

$$\mathscr{R}\theta(t) = \mathscr{R}P_t \star \theta_0 - \int_0^t \mathscr{R}P_{t-s} \star (v \cdot \nabla\theta)(s) ds \tag{6}$$

By doing the same calculation,

$$||\mathscr{R}\theta(t)||_{L^\infty} \lesssim ||\mathscr{R}\theta_0||_{L^\infty} + \int_0^t ||\mathscr{R}P_{t-s}||_{L^{\frac{4}{3}}} ||v(s)||_{L^\infty} ||\nabla\theta(s)||_{L^4} ds \tag{7}$$

By using the representation of the Oseen kernel,

$$||\mathscr{R}\theta(t)||_{L^\infty} \lesssim ||\mathscr{R}\theta_0||_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-s}} ||v(s)||_{L^\infty} ||\nabla\theta(s)||_{L^4} ds \tag{8}$$

The rest calculation is the same. In sum,

$$\begin{aligned} & ||\theta||_{L_t^\infty L^\infty} + ||v||_{L_t^\infty L^\infty} + ||\theta||_{L_t^\infty \dot{H}^1} + ||\partial^{\frac{3}{2}}\theta||_{L_t^2 L^2} \\ \lesssim\ & ||\theta_0||_{L^\infty} + ||v_0||_{L^\infty} + ||\theta_0||_{\dot{H}^1} + (||\theta||_{L_t^\infty L^\infty} + ||v||_{L_t^\infty L^\infty} + ||\theta||_{L_t^\infty \dot{H}^1} + ||\partial^{\frac{3}{2}}\theta||_{L_t^2 L^2})^2 \end{aligned} \tag{9}$$

which implies the global existence of small solution in $L_t^\infty L^\infty \cap L_t^\infty \dot{H}^1$.

3.2 Iteration

Let us define iterated system of equations. First, let $\theta^{(1)}(t,x) = e^{-t\Lambda}\theta_0$. Then, we define the system of equations inductively.

$$(DQG_{(n+1)}) \begin{cases} \theta_t^{(n+1)} + v^{(n)} \cdot \nabla\theta^{(n+1)} + \Lambda\theta^{(n+1)} = 0 \\ v^{(n)} = (-R_2\theta^{(n)}, R_1\theta^{(n)}) \\ \theta_0^{(n+1)} = \theta_0 \end{cases}$$

We define a function X with a norm $||\theta||_X$

$$||\theta||_X = ||\theta||_{L_t^\infty L^\infty} + ||v||_{L_t^\infty L^\infty} + ||\theta||_{L_t^\infty \dot{H}^1} + ||\theta||_{L_t^2 \dot{H}^{\frac{3}{2}}}$$

By a priori estimate,

$$||\theta^{(n+1)}||_X \lesssim ||\theta_0||_{L^\infty} + ||\theta^{(n)}||_X \cdot ||\theta^{(n+1)}||_X \tag{10}$$

This implies that $\{\theta^{(n)}\}$ are uniformly bounded in X for small data $\theta_0 \in L^\infty \cap \dot{H}^1, v_0 \in L^\infty$. In order to show the convergence to a function in X, we consider the evolution equations of $\theta^{(n+1)} - \theta^{(n)}$.

$$(\theta^{(n+1)} - \theta^{(n)})_t + v^{(n)} \cdot \nabla(\theta^{(n+1)} - \theta^{(n)}) + (v^{(n)} - v^{(n-1)}) \cdot \nabla\theta^{(n)} + \Lambda(\theta^{(n+1)} - \theta^{(n)}) = 0 \tag{11}$$

with zero initial data. Again, by a priori estimate, one can show that

$$||\theta^{(n+1)} - \theta^{(n)}||_X \lesssim \frac{1}{2}||\theta^{(n)} - \theta^{(n-1)}||_X \tag{12}$$

This implies the existence of a unique solution in X.

4 Proof of Corollary

Again. in this chapter, we only obtain a priori estimate in $\tilde{L}_t^\infty \dot{B}^0_{\infty,q} \cap \tilde{L}_t^1 \dot{B}^1_{\infty,q}$. First of all, we take $\triangle_j$ to the equation.

$$(\triangle_j\theta)_t + \Lambda\triangle_j\theta = -\triangle_j(v \cdot \nabla\theta) \tag{13}$$

Then, we can represent the solution $\triangle_j\theta$ in the integral form:

$$\triangle_j\theta(t) = e^{-\Lambda t}\triangle_j\theta_0 - \int_0^t e^{-\Lambda(t-s)}\triangle_j(v \cdot \nabla\theta)(s)ds \tag{14}$$

We take the L^∞ norm in the space variables. By Lemma 2.2, we have the following estimate.

$$||\triangle_j\theta(t)||_{L^\infty} \lesssim e^{-t2^j}||\triangle_j\theta_0||_{L^\infty} + \int_0^t e^{-(t-s)2^j}||\triangle_j(v \cdot \nabla\theta)(s)||_{L^\infty}ds \tag{15}$$

By taking the L^∞ norm in time, and then taking l^q for $j \in \mathbb{Z}$,

$$||\theta||_{\tilde{L}_t^\infty \dot{B}^0_{\infty,q}} \lesssim ||\theta_0||_{\dot{B}^0_{\infty,q}} + ||v \cdot \nabla\theta||_{\tilde{L}_t^1 \dot{B}^0_{\infty,q}} \tag{16}$$

By taking L^1 norm in time to (15), we obtain that

$$||\theta||_{\tilde{L}_t^\infty \dot{B}^0_{\infty,q}} + ||\theta||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} \lesssim ||\theta_0||_{\dot{B}^0_{\infty,q}} + ||v \cdot \nabla\theta||_{\tilde{L}_t^1 \dot{B}^0_{\infty,q}} \tag{17}$$

Now, we estimate the nonlinear term $||v \cdot \nabla\theta||_{\tilde{L}_t^1 \dot{B}^0_{\infty,q}}$. Here, we use the paraproduct of v and θ. We represent the product $v\theta$ by

$$v\theta = \sum_{j=-1}^{\infty} S_j v \triangle_j \theta + \sum_{j=-1}^{\infty} S_j \theta \triangle_j v + \sum_{|j-j'|\le 1} \triangle_j v \triangle_{j'} \theta \tag{18}$$

We apply the operator $\triangle_k$ to the product. Then, up to finitely many term,

$$\triangle_k(v\theta) = S_k v \triangle_k \theta + S_k \theta \triangle_k v + \sum_{j>k} \triangle_j v \triangle_j \theta \tag{19}$$

We take L^∞ norm and multiply by 2^k.

$$\begin{aligned}
& 2^k ||\triangle_k(v\theta)||_{L^\infty} \\
\lesssim\ & 2^k \Big(||S_k v||_{L^\infty} ||\triangle_k \theta||_{L^\infty} + ||S_k \theta||_{L^\infty} ||\triangle_k v||_{L^\infty} \Big) + 2^k \sum_{j>k} ||\triangle_j v||_{L^\infty} ||\triangle_j \theta||_{L^\infty} \\
\le\ & 2^k \Big(||S_k v||_{L^\infty} ||\triangle_k \theta||_{L^\infty} + ||S_k \theta||_{L^\infty} ||\triangle_k v||_{L^\infty} \Big) + 2^k \sum_{j>k} 2^{-j} 2^j ||\triangle_j v||_{L^\infty} ||\triangle_j \theta||_{L^\infty}
\end{aligned} \tag{20}$$

By taking L^1 in time,

$$\begin{aligned}
2^k ||\triangle_k(v\theta)||_{L_t^1 L^\infty} \lesssim\ & 2^k \Big(||S_k v||_{L_t^\infty L^\infty} ||\triangle_k \theta||_{L_t^1 L^\infty} + ||S_k \theta||_{L_t^\infty L^\infty} ||\triangle_k v||_{L_t^1 L^\infty} \Big) \\
+\ & 2^k \sum_{j>k} 2^{-j} 2^j ||\triangle_j v||_{L_t^\infty L^\infty} ||\triangle_j \theta||_{L_t^1 L^\infty}
\end{aligned} \tag{21}$$

Now, we use the fact that $v \in L^\infty$ and $\theta \in L^\infty$.

$$||v \cdot \nabla\theta||_{\tilde{L}_t^1 \dot{B}^0_{\infty,q}} \lesssim ||\theta||_{L_t^\infty L^\infty} ||\theta||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} + ||v||_{L_t^\infty L^\infty} ||v||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} \tag{22}$$

Since v the image of the Riesz transform,

$$||v \cdot \nabla\theta||_{\tilde{L}_t^1 \dot{B}^0_{\infty,q}} \lesssim (||\theta||_{L_t^\infty L^\infty} + ||v||_{L_t^\infty L^\infty}) ||\theta||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} \tag{23}$$

Combining (17) and (23),

$$||\theta||_{\tilde{L}_t^\infty \dot{B}^0_{\infty,q}} + ||\theta||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} \lesssim ||\theta_0||_{\dot{B}^0_{\infty,q}} + (||\theta||_{L_t^\infty L^\infty} + ||v||_{L_t^\infty L^\infty}) ||\theta||_{\tilde{L}_t^1 \dot{B}^1_{\infty,q}} \tag{24}$$

This completes a priori estimate. ■

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